

# A Universal Energy Functional for Trapped Fermi Gases with Large Scattering Length

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Yoram Alhassid conjectured that the total energy of a harmonically trapped two-component Fermi gas with large scattering length is a linear functional of the occupation probabilities of single-particle energy eigenstates. We confirm his conjecture and derive the functional explicitly. We show that the functional applies to *all smooth potentials* having a minimum, not just harmonic traps. We also calculate the occupation probabilities of high energy states.

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**Introduction**— It is well known that the energy of a noninteracting Fermi gas in a trap can be expressed as a summation over single-particle energy eigenstates

$$E = \sum_{\nu\sigma} \epsilon_\nu n_{\nu\sigma}, \quad (1)$$

where  $\nu$  and  $\sigma$  label the orbital and spin states respectively,  $\epsilon_\nu$  is the single-particle energy level, and  $n_{\nu\sigma}$  is the occupation probability of the state  $(\nu, \sigma)$ .

For interacting systems, however, there is no general relation between the total energy, which includes interaction energy, and  $n_{\nu\sigma}$  alone.

In this Letter we study a remarkable exception to this general rule, in a Fermi gas with *strong* interactions.

We consider a two-component ( $\sigma = \uparrow, \downarrow$ ) Fermi gas in which the *s*-wave scattering length  $a$  and other relevant length scales, such as the mean inter-particle spacing  $d$  and the thermal de Broglie wave length, are all much larger than the range of the interaction  $r_e \rightarrow 0$ . Applications of this *s*-wave contact interaction model range from ultracold thin vapors of neutral atoms near broad Feshbach resonances to the neutron gas with density and temperature below nuclear scales, or from extremely cold ( $10^{-9}$ - $10^{-6}$ K) to extremely hot ( $10^9$ K) matter.

The energy of such a Fermi gas was found to be [1–3]

$$E = \frac{\hbar^2 \mathcal{I}}{4\pi m a} + \int \frac{d^3 k}{(2\pi)^3} \sum_\sigma \frac{\hbar^2 k^2}{2m} \left( \rho_{\mathbf{k}\sigma} - \frac{\mathcal{I}}{k^4} \right) + \int d^3 r \sum_\sigma n_\sigma(\mathbf{r}) V(\mathbf{r}), \quad (2)$$

where  $\rho_{\mathbf{k}\sigma}$  is the momentum distribution,  $n_\sigma(\mathbf{r})$  is the spatial density distribution,  $m$  is each fermion's mass,  $V(\mathbf{r})$  is the external potential, and  $\mathcal{I} = \lim_{k \rightarrow \infty} k^4 \rho_{\mathbf{k}\sigma}$  is the *contact*, a parameter characterizing the number of small pairs of fermions [1, 2]. The contact is at the center of many universal relations for fermions with *s*-wave contact interaction [1–34].

Equation (2), containing *two continuous distributions* [ $\rho_{\mathbf{k}\sigma}$  and  $n_\sigma(\mathbf{r})$ ], is considerably more complicated than Eq. (1) which involves merely the occupation of single-particle levels. In the case of a harmonic trap,  $n_{\nu\sigma}$  are just a *discrete* set of numbers.

Yoram Alhassid conjectured that for harmonically trapped fermions with large scattering length, the total energy might still be a linear functional of  $n_{\nu\sigma}$  [35].

We will show that his conjecture is true by deriving the functional explicitly. We also show that it is valid for *all smooth potentials* having a minimum (which we set to zero without loss of generality). One can equally well apply this functional to anharmonic traps, periodic potentials (eg, an optical lattice for cold atoms), or disordered potentials, etc.

This universal energy functional is

$$E = \frac{\hbar^2 \mathcal{I}}{4\pi m a} + \lim_{\epsilon_{\max} \rightarrow \infty} \left( \sum_{\epsilon_\nu < \epsilon_{\max}} \epsilon_\nu n_\nu - \frac{\hbar \mathcal{I}}{\pi^2} \sqrt{\frac{\epsilon_{\max}}{2m}} \right), \quad (3)$$

where  $n_\nu \equiv \sum_\sigma n_{\nu\sigma}$ . The contact  $\mathcal{I}$  is contained in the asymptotic behavior of

$$\rho_\sigma(\epsilon) \equiv \sum_\nu n_{\nu\sigma} \delta(\epsilon - \epsilon_\nu) \quad (4)$$

at high energy,

$$\rho_\sigma(\epsilon)|_{\text{coarse grained}} = \frac{\hbar \mathcal{I}}{4\pi^2 \sqrt{2m}} \epsilon^{-3/2} + O(\epsilon^{-5/2}). \quad (5)$$

We also calculate the occupation probabilities of individual single-particle energy eigenstates at  $\epsilon_\nu \gg \max\{|E|/N, \hbar^2/m a^2, \hbar^2/m d^2, \Delta V\}$ , where  $N$  is the number of fermions, and  $\Delta V$  is the characteristic range of potentials involved in the  $N$ -body state. (For example, if many fermions form a cloud in a trap,  $\Delta V$  is the change of  $V(\mathbf{r})$  from the trap minimum to the edge of the cloud.) The result is

$$n_{\nu\sigma} = \frac{1}{k_\nu^4} \int C(\mathbf{r}) |\phi_\nu(\mathbf{r})|^2 d^3 r + \frac{4m^2}{\hbar^2 k_\nu^6} \int \mathbf{D}(\mathbf{r}) \cdot \mathbf{j}_\nu(\mathbf{r}) d^3 r + O(\epsilon_\nu^{-3}), \quad (6)$$

where  $k_\nu \equiv \sqrt{2m\epsilon_\nu}/\hbar$ ,  $C(\mathbf{r})$  is the contact density [1, 2] [related to the total contact:  $\int C(\mathbf{r}) d^3 r = \mathcal{I}$ ],  $\phi_\nu(\mathbf{r})$  is the normalized wave function of the  $\nu$ -th single particle orbital, satisfying the Schrödinger equation

$$[-(\hbar^2/2m)\nabla^2 + V(\mathbf{r})]\phi_\nu(\mathbf{r}) = \epsilon_\nu \phi_\nu(\mathbf{r}), \quad (7)$$

$\mathbf{D}(\mathbf{r})$  is the “contact current” [see Eq. (17) below], and  $\mathbf{j}_\nu(\mathbf{r}) \equiv (\hbar/m) \text{Im} \phi_\nu^* \nabla \phi_\nu \sim O[(\epsilon_\nu/m)^{1/2} |\phi_\nu|^2]$  is the probability current of the  $\nu$ -th single-particle orbital state.

Equations (3), (5) and (6) apply to both energy eigenstates and thermal ensembles, both equilibrium and non-equilibrium states, both few-body and many-body systems, both strong ( $|a| \gtrsim d$ ) and weak ( $|a| \ll d$ ) interactions, both symmetric ( $N_\uparrow = N_\downarrow$ ) and polarized ( $N_\uparrow \neq N_\downarrow$ ) states. In the many-body regime, they are valid for *all phases*, including normal and superfluid phases.

In the following we first derive an expansion for the one-particle density matrix  $p_\sigma(\mathbf{r}, \mathbf{r} + \mathbf{b}) \equiv \langle \psi_\sigma^\dagger(\mathbf{r}) \psi_\sigma(\mathbf{r} + \mathbf{b}) \rangle$  at a small separation  $\mathbf{b}$ . From this we derive Eq. (3) (exploiting the propagator of a single particle in a short imaginary time), Eq. (5), and Eq. (6). The derivations are for energy eigenstates but can be easily extended to thermal ensembles and non-equilibrium states.

**One-Particle Density Matrix**— Consider a normalized  $N$ -body energy eigenstate ( $N = N_\uparrow + N_\downarrow$ )

$$|\phi\rangle = (N_\uparrow! N_\downarrow!)^{-1/2} \int D_1^\uparrow D_1^\downarrow \phi(\mathbf{r}_1, \dots, \mathbf{r}_{N_\uparrow}, \mathbf{s}_1, \dots, \mathbf{s}_{N_\downarrow}) \times \psi_\uparrow^\dagger(\mathbf{r}_1) \dots \psi_\uparrow^\dagger(\mathbf{r}_{N_\uparrow}) \psi_\downarrow^\dagger(\mathbf{s}_1) \dots \psi_\downarrow^\dagger(\mathbf{s}_{N_\downarrow}) |0\rangle, \quad (8)$$

where  $|0\rangle$  is the particle vacuum,  $\psi_\sigma^\dagger(\mathbf{r})$  is the standard fermion creation operator, and we have introduced shorthand notations  $D_i^\uparrow \equiv \prod_{\mu=i}^{N_\uparrow} d^3 r_\mu$  and  $D_i^\downarrow \equiv \prod_{\mu=i}^{N_\downarrow} d^3 s_\mu$ . When  $\mathbf{r}_1$  and  $\mathbf{s}_1$  are close,  $\phi(\mathbf{r}_1, \dots, \mathbf{r}_{N_\uparrow}, \mathbf{s}_1, \dots, \mathbf{s}_{N_\downarrow})$  satisfies the Bethe-Peierls boundary condition

$$\phi = A(\tfrac{1}{2}(\mathbf{r}_1 + \mathbf{s}_1); \mathbf{r}_2 \dots \mathbf{r}_{N_\uparrow}, \mathbf{s}_2 \dots \mathbf{s}_{N_\downarrow}) (|\mathbf{r}_1 - \mathbf{s}_1|^{-1} - a^{-1}) + O(|\mathbf{r}_1 - \mathbf{s}_1|). \quad (9)$$

We now expand the 1-particle density matrix

$$p_\uparrow(\mathbf{r}, \mathbf{r} + \mathbf{b}) = \langle \phi | \psi_\uparrow^\dagger(\mathbf{r}) \psi_\uparrow(\mathbf{r} + \mathbf{b}) | \phi \rangle = N_\uparrow \int D_2^\uparrow D_1^\downarrow \phi^*(\mathbf{r}, \mathbf{r}_2 \dots \mathbf{r}_{N_\uparrow}, \mathbf{s}_1 \dots \mathbf{s}_{N_\downarrow}) \phi(\mathbf{r} + \mathbf{b}, \mathbf{r}_2 \dots) \quad (10)$$

through order  $O(b^2)$  at a small distance  $b$ . Because of the singularity of  $\phi$  when two fermions in different spin states are close (see above), we divide the  $3(N_\uparrow + N_\downarrow - 1)$  dimensional integration domain into region  $\mathcal{R}_\epsilon$  (in which *every* spin-down fermion lies outside of the sphere of radius  $\epsilon$  centered at  $\mathbf{r}$ , namely  $|\mathbf{s}_\mu - \mathbf{r}| > \epsilon$  for  $\mu = 1, \dots, N_\downarrow$ ) and its complement,  $\overline{\mathcal{R}_\epsilon}$ . Here  $\epsilon$  is small but  $\epsilon > b$ . In  $\mathcal{R}_\epsilon$  we expand  $\phi(\mathbf{r} + \mathbf{b}, \mathbf{r}_2 \dots \mathbf{r}_{N_\uparrow}, \mathbf{s}_1 \dots \mathbf{s}_{N_\downarrow})$  in powers of  $\mathbf{b}$ , while in  $\overline{\mathcal{R}_\epsilon}$  we use Eq. (9) which is sufficient for evaluating the integral in  $\overline{\mathcal{R}_\epsilon}$  through order  $b^2$ . In  $\overline{\mathcal{R}_\epsilon}$  it is possible for two or more spin-down fermions to come inside the small sphere of radius  $\epsilon$  centered at  $\mathbf{r}$ , but the contributions from such cases are suppressed by Fermi statistics and are of higher order than  $O(b^3)$ . When the integrals in

the two regions  $\mathcal{R}_\epsilon$  and  $\overline{\mathcal{R}_\epsilon}$  are added, all dependencies on  $\epsilon$  are canceled, yielding the following clean expansion:

$$p_\uparrow(\mathbf{r}, \mathbf{r} + \mathbf{b}) = n_\uparrow(\mathbf{r}) + C(\mathbf{r})(-b/8\pi + b^2/24\pi a) + \mathbf{b} \cdot \mathbf{u}_\uparrow(\mathbf{r}) - 3\pi b \mathbf{b} \cdot \mathbf{w}(\mathbf{r})/2 - \pi b \mathbf{b} \cdot \mathbf{w}^*(\mathbf{r})/2 + \sum_{i,j=1}^3 v_{\uparrow ij}(\mathbf{r}) b_i b_j/2 + O(b^3), \quad (11)$$

where  $b_i$  is the  $i$ -th Cartesian component of  $\mathbf{b}$ ,

$$n_\uparrow(\mathbf{r}) = N_\uparrow \int D_2^\uparrow D_1^\downarrow |\phi(\mathbf{r}, \mathbf{r}_2 \dots \mathbf{r}_{N_\uparrow}, \mathbf{s}_1 \dots \mathbf{s}_{N_\downarrow})|^2 \quad (12)$$

is the density of spin-up fermions at position  $\mathbf{r}$ ,

$$C(\mathbf{r}) = 16\pi^2 N_\uparrow N_\downarrow \int D_2^\uparrow D_2^\downarrow |A(\mathbf{r}; \mathbf{r}_2 \dots \mathbf{r}_{N_\uparrow}, \mathbf{s}_2 \dots \mathbf{s}_{N_\downarrow})|^2 \quad (13)$$

is the contact density [1, 2],

$$\mathbf{w}(\mathbf{r}) \equiv N_\uparrow N_\downarrow \int D_2^\uparrow D_2^\downarrow A^*(\mathbf{r}; \mathbf{r}_2 \dots \mathbf{r}_{N_\uparrow}, \mathbf{s}_2 \dots \mathbf{s}_{N_\downarrow}) \times \nabla_r A(\mathbf{r}; \mathbf{r}_2 \dots \mathbf{r}_{N_\uparrow}, \mathbf{s}_2 \dots \mathbf{s}_{N_\downarrow}) \quad (14)$$

is related to the center-of-mass motion of small pairs, and

$$\mathbf{u}_\uparrow(\mathbf{r}) \equiv N_\uparrow \lim_{\eta \rightarrow 0} \int_{\mathcal{R}_\eta} D_2^\uparrow D_1^\downarrow \phi^*(\mathbf{r}, \mathbf{r}_2 \dots \mathbf{r}_{N_\uparrow}, \mathbf{s}_1 \dots \mathbf{s}_{N_\downarrow}) \times \nabla_r \phi(\mathbf{r}, \mathbf{r}_2 \dots \mathbf{r}_{N_\uparrow}, \mathbf{s}_1 \dots \mathbf{s}_{N_\downarrow}), \quad (15)$$

$$v_{\uparrow ij}(\mathbf{r}) \equiv N_\uparrow \lim_{\eta \rightarrow 0} \int_{\mathcal{R}_\eta} D_2^\uparrow D_1^\downarrow \phi^*(\mathbf{r}, \mathbf{r}_2 \dots \mathbf{r}_{N_\uparrow}, \mathbf{s}_1 \dots \mathbf{s}_{N_\downarrow}) \times \frac{\partial^2}{\partial r_i \partial r_j} \phi(\mathbf{r}, \mathbf{r}_2 \dots \mathbf{r}_{N_\uparrow}, \mathbf{s}_1 \dots \mathbf{s}_{N_\downarrow}). \quad (16)$$

There is of course a completely analogous expansion for  $p_\downarrow(\mathbf{r}, \mathbf{r} + \mathbf{b})$  involving the same  $C(\mathbf{r})$  and  $\mathbf{w}(\mathbf{r})$ .

In addition to the contact density  $C(\mathbf{r})$  [1, 2], we introduce a “contact current”

$$\mathbf{D}(\mathbf{r}) \equiv (8\pi^2 \hbar/m) \text{Im} \mathbf{w}(\mathbf{r}). \quad (17)$$

There is in general no continuity relation between the contact density and the contact current, because the small pairs may dissociate or associate.

**Universal Energy Functional**— For any  $(N_\uparrow + N_\downarrow)$ -body energy eigenstate  $|\phi\rangle$  and any  $\beta$  satisfying  $\text{Re} \beta \geq 0$  we define an absolutely convergent series:

$$J_\sigma(\beta) \equiv \sum_\nu n_{\nu\sigma} e^{-\beta \epsilon_\nu} = \sum_\nu \langle \phi | c_{\nu\sigma}^\dagger c_{\nu\sigma} | \phi \rangle e^{-\beta \epsilon_\nu}. \quad (18)$$

Since the fermion annihilation operator

$$c_{\nu\sigma} = \int d^3 r \phi_\nu^*(\mathbf{r}) \psi_\sigma(\mathbf{r}), \quad (19)$$

we have

$$J_\sigma(\beta) = \int d^3r d^3r' U_\beta(\mathbf{r}, \mathbf{r}') p_\sigma(\mathbf{r}, \mathbf{r}'), \quad (20)$$

where  $U_\beta(\mathbf{r}, \mathbf{r}') \equiv \sum_\nu e^{-\beta\epsilon_\nu} \phi_\nu(\mathbf{r}) \phi_\nu^*(\mathbf{r}')$  is the propagator of a single particle moving in the potential  $V(\mathbf{r})$  within a time  $-i\hbar\beta$ . For a small positive  $\beta$ , at  $|\mathbf{r} - \mathbf{r}'| \gg \hbar\sqrt{\beta/m}$  the propagator is exponentially suppressed, while at  $|\mathbf{r} - \mathbf{r}'| \sim \hbar\sqrt{\beta/m}$  we have a “short imaginary-time expansion”

$$U_\beta(\mathbf{r}, \mathbf{r}') = (2\pi\hbar^2\beta/m)^{-3/2} \{1 - [V(\mathbf{r}) + V(\mathbf{r}')]\beta/2\} \\ \times \exp[-m(\mathbf{r} - \mathbf{r}')^2/2\hbar^2\beta] + O(\beta^{1/2}), \quad (21)$$

provided that  $V(\mathbf{r})$  is smooth. But when  $|\mathbf{r} - \mathbf{r}'|$  is small we also have a systematic expansion for  $p_\sigma(\mathbf{r}, \mathbf{r}')$  [see above]. Substituting both expansions into Eq. (20) we obtain a systematic expansion for  $J_\sigma(\beta)$  at small  $\beta$ :

$$J_\sigma(\beta) = N_\sigma - (\hbar\mathcal{I}/4\pi^2) \sqrt{2\pi\beta/m} + \hbar^2\mathcal{I}\beta/8\pi ma \\ - \beta \int d^3r V(\mathbf{r}) n_\sigma(\mathbf{r}) \\ + (\hbar^2\beta/2m) \int d^3r \sum_{i=1}^3 v_{\sigma ii}(\mathbf{r}) + O(\beta^{3/2}). \quad (22)$$

From the  $N$ -body Schrödinger equation

$$\left\{ \sum_{\mu=1}^{N_\uparrow} \left[ -\frac{\hbar^2}{2m} \nabla_{\mathbf{r}_\mu}^2 + V(\mathbf{r}_\mu) \right] + \sum_{\mu'=1}^{N_\downarrow} \left[ -\frac{\hbar^2}{2m} \nabla_{\mathbf{s}_{\mu'}}^2 + V(\mathbf{s}_{\mu'}) \right] \right\} \phi \\ = E\phi, \quad \text{if } \mathbf{r}_\mu \neq \mathbf{s}_{\mu'} \text{ for all } \mu, \mu', \quad (23)$$

one can show that

$$\sum_\sigma \int d^3r \left[ V(\mathbf{r}) n_\sigma(\mathbf{r}) - \frac{\hbar^2}{2m} \sum_{i=1}^3 v_{\sigma ii}(\mathbf{r}) \right] = E, \quad (24)$$

so the summation of Eq. (22) over  $\sigma$  yields

$$\sum_{\nu\sigma} n_{\nu\sigma} e^{-\beta\epsilon_\nu} = N - \frac{\hbar\mathcal{I}}{2\pi^2} \sqrt{\frac{2\pi\beta}{m}} + \frac{\hbar^2\mathcal{I}\beta}{4\pi ma} - \beta E + O(\beta^{3/2}). \quad (25)$$

Applying  $\frac{d}{d\beta}$  to the above expansion, defining  $\rho(\epsilon) = \sum_{\nu\sigma} n_{\nu\sigma} \delta(\epsilon - \epsilon_\nu)$ , and taking  $\beta \rightarrow 0$ , we find

$$E = \frac{\hbar^2\mathcal{I}}{4\pi ma} + \lim_{\beta \rightarrow 0} \int_0^\infty \left[ \rho(\epsilon) - \frac{\hbar\mathcal{I}}{2\pi^2\sqrt{2m}} \epsilon^{-3/2} \right] \epsilon e^{-\beta\epsilon} d\epsilon \\ = \frac{\hbar^2\mathcal{I}}{4\pi ma} + \int_0^\infty \left[ \rho(\epsilon) - \frac{\hbar\mathcal{I}}{2\pi^2\sqrt{2m}} \epsilon^{-3/2} \right] \epsilon d\epsilon, \quad (26)$$

and thus Eq. (3).

**Asymptotics of  $\rho_\sigma(\epsilon)$ –** When  $\beta = it/\hbar$  is purely imaginary and the real “time”  $t \rightarrow \pm 0$ , the net contribution to the integral in Eq. (20) from  $|\mathbf{r} - \mathbf{r}'| \gg \sqrt{\hbar|t|/m}$  is exponentially small because of the rapid oscillation of

the propagator  $U$ . When  $|\mathbf{r} - \mathbf{r}'| \sim \sqrt{\hbar|t|/m}$  the expansion in Eq. (21) with  $\beta$  replaced by  $it/\hbar$  holds [36]. Therefore,  $J_\sigma(it/\hbar)$  has a “short real-time expansion” by simply setting  $\beta = it/\hbar$  in Eq. (22), and thus has an  $O(\sqrt{|t|})$  singularity at  $t = 0$ . So the function  $\rho_\sigma(\epsilon)$  defined in Eq. (4), which is the Fourier transform of  $J_\sigma$ :

$$\rho_\sigma(\epsilon) = (2\pi\hbar)^{-1} \int_{-\infty}^\infty J_\sigma(it/\hbar) e^{i\epsilon t/\hbar} dt, \quad (27)$$

has a “coarse-grained” asymptotic formula at high energy shown in Eq. (5). For a deep trap, the fact that  $\rho_\sigma(\epsilon)$  remains a discrete sum of delta functions at large  $\epsilon$ , rather than turning into a continuous curve, can be traced to the singularities of  $J_\sigma(it/\hbar)$  at nonzero  $t$ 's. But in a “coarse grained” distribution,  $\rho_\sigma(\epsilon)|_{\text{coarse grained}} = \int_{-\infty}^\infty g(\epsilon') \rho_\sigma(\epsilon - \epsilon') d\epsilon'$ , where the convolution factor may be chosen as  $g(\epsilon') = \exp(-\epsilon'^2/\lambda^2)/(\lambda\sqrt{\pi})$  with a large width  $\lambda$ , these singularities are “washed out”, because  $\rho_\sigma(\epsilon)|_{\text{coarse grained}}$  is the Fourier transform of the product of  $\tilde{g}(t)$  and  $J_\sigma(it/\hbar)$ , where  $\tilde{g}(t)$  is the inverse Fourier transform of  $g(\epsilon)$  and decays exponentially at  $|t| \gg \hbar/\lambda$ . For the validity of Eq. (5), the energy resolution  $\lambda$  should not grow faster than  $\text{constant} \times \sqrt{\epsilon}$ .

**Asymptotics of  $n_{\nu\sigma}$ –** From Eq. (19) we find

$$n_{\nu\sigma} = \int d^3r \phi_\nu(\mathbf{r}) \int d^3b \phi_\nu^*(\mathbf{r} + \mathbf{b}) p_\sigma(\mathbf{r}, \mathbf{r} + \mathbf{b}). \quad (28)$$

At large  $\epsilon_\nu$ , the integrand as a function of  $\mathbf{b}$  oscillates rapidly. The only significant contribution comes from the power-law singularities of  $p_\sigma$  at  $\mathbf{b} \rightarrow 0$ . According to Eq. (11), the leading order singular term is  $\propto |\mathbf{b}|$ , for which we write  $\phi_\nu^*(\mathbf{r} + \mathbf{b}) \doteq (-\hbar^2/2m\epsilon_\nu)^2 \nabla_b^4 \phi_\nu^*(\mathbf{r} + \mathbf{b})$  with relative error  $\sim O(\epsilon_\nu^{-1})$  according to Eq. (7) [37]. Integration by parts over  $\mathbf{b}$  yields  $\propto \int d^3b \phi_\nu^*(\mathbf{r} + \mathbf{b}) \delta(\mathbf{b})$ , leading to the first term on the right hand side of Eq. (6).

The next order singular term in  $p_\sigma$  is  $\propto b\mathbf{b}$ , for which we write  $\phi_\nu^*(\mathbf{r} + \mathbf{b}) \doteq (-\hbar^2/2m\epsilon_\nu)^3 \nabla_b^6 \phi_\nu^*(\mathbf{r} + \mathbf{b})$  with relative error  $\sim O(\epsilon_\nu^{-1})$ . Integration by parts over  $\mathbf{b}$  yields  $\propto \int d^3b \phi_\nu^*(\mathbf{r} + \mathbf{b}) \nabla_b \delta(\mathbf{b}) = -\nabla \phi_\nu^*(\mathbf{r})$ . Further integrating by parts over  $\mathbf{r}$ , omitting contributions  $\sim O(\epsilon_\nu^{-3})$ , and using Eq. (17) and the identities  $\text{Re } \mathbf{w}(\mathbf{r}) = \nabla C(\mathbf{r})/32\pi^2$  and  $\nabla \cdot \mathbf{j}_\nu(\mathbf{r}) = 0$ , we obtain the second term on the right hand side of Eq. (6).

Because any single-particle orbital state and its time reversal have the same energy but opposite probability currents, the second term on the right hand side of Eq. (6) has no net contribution to the distribution  $\rho_\sigma(\epsilon)$ .

We now illustrate Eq. (6) with a symmetric unitary ( $|k_F a| \gg 1$ ) Fermi gas at zero temperature, confined by a spherical harmonic trap of angular frequency  $\omega$ . At large  $N$  the local density approximation (LDA) for the contact density is valid:  $C(r) = C_1 k_{F0}^4 (1 - r^2/R^2)^2$ . Here  $k_{F0}$  is the local Fermi wave number at the trap center,  $R$  is the LDA cloud radius, and  $C_1 k_F^4$  is the contact density of the homogeneous unitary Fermi gas with Fermi wave number

$k_F$ . For a high energy orbital  $\nu = (j, l, m_z)$  with energy  $\epsilon_\nu = (j + 3/2)\hbar\omega$ , orbital angular momentum quantum number  $l$ , and magnetic quantum number  $m_z$  we find [38]

$$n_{\nu\sigma} \doteq \frac{16C_1 N^{5/6}}{5\pi 3^{1/6} \xi^{3/4} j^{5/2}} \left[ 1 - \frac{l(l+1)}{4\sqrt{\xi}(3N)^{1/3}j} \right]^{5/2} \quad (29)$$

if  $l(l+1) < 4\sqrt{\xi}(3N)^{1/3}j$ . For higher  $l$ , the classical orbit is outside of the LDA cloud radius, and  $n_{\nu\sigma}$  becomes exponentially small. Here  $\xi$  is the Bertsch parameter [39], ie the ratio between the ground state energy of the unitary Fermi gas and that of the noninteracting Fermi gas at the same density. In latest numerical and experimental studies,  $\xi \lesssim 0.38$  [40] and  $C_1 \approx 0.12$  [32–34, 41].

To conclude, we have shown that the total energy of fermions with large scattering length ( $|a| \gg r_e$ ) in any smooth potential is a simple linear functional of the occupation probabilities of single-particle energy eigenstates. We have also derived asymptotic expressions for the occupation probabilities of high energy states. These results can be verified experimentally by measuring the energy and the occupation probabilities independently. They also provide robust constraints on theories of trapped Fermi gases, including fermionic atoms in optical lattices. These results can be extended to lower dimensions, to fermions with unequal masses, and to bosons and Bose-Fermi mixtures.

The universal energy functional Eq. (3) implies a new approach to the difficult many-body problem at large scattering length: by identifying nontrivial constraints on the occupation probabilities, one can minimize the functional to find the many-body ground state energies in external potentials.

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[36] For certain potentials with steep edges, such as  $V(\mathbf{r}) \propto x^{2s} + y^{2s} + z^{2s}$  (where  $s = 2, 3, 4, \dots$ ), there are additional, highly oscillatory, contributions to  $U$  at small  $t$  and  $|\mathbf{r} - \mathbf{r}'|$  from energetic reflections by the edges. Their contributions to  $J_\sigma(it/\hbar)$  are of higher order than  $O(|t|^{3/2})$ .  
[37] Although in the entire space  $V(\mathbf{r})$  may not have an upper bound, in the region where  $p_\sigma(\mathbf{r}, \mathbf{r} + \mathbf{b})$  is *not* exponentially small,  $V(\mathbf{r} + \mathbf{b})$  is finite and hence  $V(\mathbf{r} + \mathbf{b})/\epsilon_\nu \rightarrow 0$  when  $\epsilon_\nu \rightarrow \infty$ .  
[38] Here  $\int C(\mathbf{r})|\phi_\nu(\mathbf{r})|^2 d^3r \approx$  the *time*-average of the contact density as sampled by a particle in the *classical* orbit.  
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